ON THE CLASSIFICATION OF GALOIS OBJECTS OVER THE QUANTUM GROUP OF A NONDEGENERATE BILINEAR FORM

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Abstract. We study Galois and bi-Galois objects over the quantum group of a nondegenerate bilinear form, including the quantum group $\mathcal{O}_q(SL(2))$. We obtain the classification of these objects up to isomorphism and some partial results for their classification up to homotopy.

Introduction

Hopf-Galois extensions and objects are quantum analogues of principal fibre bundles and torsors. It is in general a difficult problem to classify these objects. Several authors have already contributed to this problem, mainly in the finite dimensional case ([Ma1], [Ma2], [Sa2], [PO], [Bi2]...). In this paper we study a very different class of infinite dimensional Hopf algebras, including the quantum group $\mathcal{O}_q(SL(2))$ of functions over SL_2. We obtain the classification up to isomorphism and present some partial results for the classification up to homotopy. Homotopy for Hopf-Galois extensions was introduced by Kassel [K1] and developped with Schneider [KS] in order to classify Galois extensions up to a coarser equivalence relation than isomorphism. This relation is very useful for pointed Hopf algebras but it appears that, when the Hopf algebra is the quantum group $\mathcal{O}_q(SL(2))$, the classification up to homotopy is harder to obtain than the one up to isomorphism.

We consider the Hopf algebras $\mathcal{B}(E)$ introduced by Dubois-Violette and Launer [DL] as the quantum groups of nondegenerate bilinear forms given by invertible matrices E over a field k. One simple and interesting example of such a Hopf algebra is the quantum group $\mathcal{O}_q(SL(2))$ of functions over SL_2. Bichon [Bi1] has proved that the representation category of each Hopf algebra $\mathcal{B}(E)$ is monoidally equivalent to the one of $\mathcal{O}_q(SL(2))$, where q is a solution of the equation

$$q^2 + \mathrm{Tr}(E^{-1}E^t)q + 1 = 0.$$

The main ingredient of his proof is the construction of a $\mathcal{B}(E)$-$\mathcal{O}_q(SL(2))$-bi-Galois object $\mathcal{B}(E, E_q)$ for a well-chosen invertible matrix E_q. In fact, such Galois objects $\mathcal{B}(E, F)$ can be defined even when k is only assumed to be a commutative ring. They are generic in the following sense: if k is a PID (principal ideal domain), for any $\mathcal{B}(E)$-Galois object Z there exist an integer $m \geq 2$

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and an invertible matrix $F \in GL_m(k)$ such that Z is isomorphic to $\mathcal{B}(E, F)$. Moreover, two such $\mathcal{B}(E)$-Galois objects $\mathcal{B}(E, F_1)$ and $\mathcal{B}(E, F_2)$ are isomorphic if and only if there exists $P \in GL(k)$ such that $F_1 = PF_2P^t$ (P^t denotes the transpose of P). In the case when k is a field, we obtain a full classification up to isomorphism of the Galois objects of $B(E)$. As a consequence, the group of $\mathcal{B}(E)$-bi-Galois objects is trivial as well as the lazy cohomology group. Note that Ostrik [O] has recently classified module categories over representations of $\mathcal{O}_q(SL(2))$, which together with Ulbrich's and Schauenburg's work (see [U1], [U2] and [Sa3]) also yields a classification of Galois objects, but the tools used in [O] are very different from ours.

Concerning the classification up to homotopy, we prove a partial result. Namely, we show that two Galois objects $\mathcal{B}(E, F_1)$ and $\mathcal{B}(E, F_2)$ are homotopically equivalent if the matrices $F_1^{-1}F_1^t$ and $F_2^{-1}F_2^t$ have the same characteristic polynomial. In particular, any cleft $\mathcal{O}_q(SL(2))$-Galois object is homotopically trivial.

The paper is organized as follows. In Section 1 we recall some basic facts on Galois and bi-Galois extensions. Section 2 and 3 are devoted to the isomorphism problem for $\mathcal{B}(E)$-Galois objects, while Section 4 deals with the classification up to homotopy.

1. Hopf-Galois extensions and bi-Galois objects

Let k be a commutative ring. All objects in this paper belong to the tensor category of k-modules and the tensor product over k is denoted by $\otimes$. Let H be a Hopf algebra and Z be a left H-comodule algebra with coaction $\delta : Z \to H \otimes Z$. We define the subalgebra $R = Z^{coH}$ of H**-coinvariant elements** of Z by

$$R = \{z \in Z \mid \delta(z) = 1 \otimes z\}.$$

The linear application $\mathrm{can} : Z \otimes_R Z \to H \otimes Z$ given by

$$\mathrm{can}(z \otimes z') = \delta(z)(1 \otimes z')$$

for all z, $z' \in Z$, is called the **canonical map** of Z.

If Z is a left H-comodule algebra and R is a subalgebra of Z, then we say that $R \subset Z$ is a H**-Galois extension** if the subalgebra of H-coinvariant elements is R and if the canonical map $\mathrm{can} : Z \otimes_R Z \to H \otimes Z$ of Z is an isomorphism. In this case, we also say that Z is an H-Galois extension of R. A Galois extension Z of R is said to be **faithfully flat** if Z is faithfully flat as a right or left R-module. An H**-Galois object** is an H-Galois extension of k which is k-faithfully flat.

A **morphism of Galois extensions** between two H-Galois extensions Z and Z' of R is a morphism of H-comodule algebras which is the identity on R. If Z' is faithfully flat, it is always an isomorphism. We denote $\mathrm{Gal}_R(H/k)$ the set of isomorphism classes of faithfully flat H-Galois extensions of R. If Z is a faithfully flat H-Galois extension of R, its isomorphism class in $\mathrm{Gal}_R(H/k)$ is denoted by $[Z]$. If one of the objects R or k is clear, we will omit it from the notation. In the same way, one can define right H-Galois extensions of R and

we denote $\mathrm{Gal}^r_R(H/k)$ the set of isomorphism classes of faithfully flat right H-Galois extensions. If H has a bijective antipode, then there is a bijection between the sets $\mathrm{Gal}_R(H/k)$ and $\mathrm{Gal}^r_R(H/k)$.

Recall that, if H is a Hopf algebra, U is a right H-comodule and V a left H-comodule, the **cotensor product** $U\square_H V$ is the kernel of the map

$$\delta_U \otimes \mathrm{id}_V - \mathrm{id}_U \otimes \delta_V : U \otimes V \to U \otimes H \otimes V,$$

(or the equalizer of the coactions of U and V).

A bilinear map $\sigma : H \times H \to k$ is a **right invertible cocycle** for the Hopf algebra H if σ is convolution-invertible and satisfies the relations

$$\sigma(x_{(1)}y_{(1)}, z)\sigma(x_{(2)}, y_{(2)}) = \sigma(x, y_{(1)}z_{(1)})\sigma(y_{(2)}, z_{(2)})$$

and

$$\sigma(1, x) = \sigma(x, 1) = \varepsilon(x),$$

for all $x, y, z \in H$. Here ε denotes the counit of H and we have used Sweedler's notation $x_{(1)} \otimes x_{(2)}$ for the comultiplication. Note that we use right cocycles whose definition is different from the one of left cocycles (see [Mo]). We denote σ^{-1} the inverse of σ for the convolution ; σ^{-1} is a left cocycle.

Recall ([Mo, Chapter 7]) that if H is a Hopf algebra, $\sigma : H \times H \to k$ an invertible cocycle, one can define the Hopf algebra H^σ as the coalgebra H with the twisted product

$$x \cdot_\sigma y = \sigma^{-1}(x_{(1)}, y_{(1)})x_{(2)}y_{(2)}\sigma(x_{(3)}, y_{(3)})$$

and the H-comodule algebra H_σ as the left H-comodule H with the twisted product

$$x \cdot_\sigma y = x_{(1)}y_{(1)}\sigma(x_{(2)}, y_{(2)}),$$

for any $x, y \in H$. The H-comodule algebra H_σ is an H-Galois extension of k and all such Galois extensions are called **cleft Galois extensions**. If H is k-faithfully flat, it is a **cleft Galois object**.

Kassel and Schneider [KS] (see also [K1]) have defined an equivalence relation denoted $\sim$ and called **homotopy** on the class of faithfully flat Galois extensions of R. Two Hopf-Galois extensions are homotopy equivalent if there exists a polynomial path between these extensions. More precisely, let $k[t]$ be the algebra of polynomials with coefficients in the ground ring k. For any k-module V, we denote $V[t] = V \otimes k[t]$ and for $i \in \{0, 1\}$ we denote $[i] : V[t] \to V$ the k-linear map sending vt^n to vi^n. These two maps $[i]$ induce two maps

$$[i]_* : \mathrm{Gal}_{R[t]}(H[t], k[t]) \to \mathrm{Gal}_R(H, k),$$

for $i = 0, 1$. We say that two H-Galois extensions Z_0 and $Z_1 \in \mathrm{Gal}_R(H/k)$ are **homotopy equivalent** if there exists $Z \in \mathrm{Gal}_{R[t]}(H[t]/k[t])$ such that $[i]_*(Z) = Z_i$ for $i \in \{0, 1\}$. We denote $\mathcal{H}_R(H)$ the set of homotopy classes of faithfully flat left H-Galois extensions of R.

Kassel and Schneider [KS, Proposition 1.6, Corollary 1.11] have proved that twists of homotopy equivalent Galois objects are still homotopy equivalent. In fact, the twist is a particular case of the cotensor product by a bi-Galois object. We generalize this result now.

Let H and K be Hopf algebras. An **H-K-bi-Galois object** is a H-K-bicomodule algebra Z which is a Galois object with respect to the right and the left coactions. By work of Schauenburg [Sa1] (see also [Sa3]), the set of bi-Galois objects is a groupoid with the multiplication given by the cotensor product. In particular, when $H = K$, the cotensor product over $H = K$ puts a structure of group on the set of isomorphism classes of H-H-bi-Galois objects. If Z is an H-K-bi-Galois object, the cotensor product yields a bijective map $\varphi_Z : \mathrm{Gal}_k(K) \to \mathrm{Gal}_k(H)$ defined by

$$\varphi_Z([A]) = [Z \square_K A]$$

for any left K-Galois object A (see [Sa1] and [Sa3] for details).

Proposition 1. *For any H-K-bi-Galois object Z, the map φ_Z induces a bijective map $\overline{\varphi_Z} : \mathcal{H}_k(K) \to \mathcal{H}_k(H)$ between the homotopy classes of left K-Galois objects and of left H-Galois objects.*

Proof. Let A_0, A_1 be homotopically equivalent H-Galois objects via the $H[t]$-Galois object A. Then $Z[t] = Z \otimes k[t]$ is an $H[t]$-$K[t]$-bi-Galois object and $Z[t]\square_{K[t]}A$ is an homotopy between $Z\square_K A_0$ and $Z\square_K A_1$. There exists a K-H-bi-Galois object Z^{-1} inverse of Z for the groupoid structure of bi-Galois objects and the map $\overline{\varphi_{Z^{-1}}} : \mathcal{H}(H) \to \mathcal{H}(K)$ induced by Z^{-1} is the inverse of the map $\overline{\varphi_Z} : \mathcal{H}(K) \to \mathcal{H}(H)$ induced by Z. $\square$

2. The Hopf algebra $\mathcal{B}(E)$ and the comodule algebra $\mathcal{B}(E, F)$

Let k be a commutative ring, $n \geq 1$ an integer and $E = (E_{ij})_{1\leq i,j\leq n} \in GL_n(k)$. Following [DL], we define $\mathcal{B}_k(E)$ (or $\mathcal{B}(E)$ when the base ring is clear) as the k-algebra generated by n^2 variables a_{ij}, $1 \leq i, j \leq n$, submitted to the matrix relations

$$E^{-1}a^tEa = \mathrm{I_n} = aE^{-1}a^tE,$$

where E^{-1} is the inverse matrix of E, a is the matrix (a_{ij}), $\mathrm{I_n}$ the identity matrix of size n and a^t denotes the transpose of the matrix a.

The algebra $\mathcal{B}(E)$ is a Hopf algebra with comultiplication Δ defined by

$$\Delta(a_{ij}) = \sum_{k=1}^{n} a_{ik} \otimes a_{kj},$$

counit ε defined by $\varepsilon(a_{ij}) = \delta_{ij}$, for any $i, j = 1, \dots, n$, where δ_{ij} is Kronecker's symbol, and antipode S defined by the matrix identity $S(a) = E^{-1}a^tE$.

Note that if $n = 1$, the Hopf algebra $\mathcal{B}(E)$ is isomorphic to $k[\mathbb{Z}/2\mathbb{Z}]$, whose Galois objects are $k[\mathbb{Z}/2\mathbb{Z}]_\sigma$ for $\sigma \in H^2(\mathbb{Z}/2\mathbb{Z}, k^*)$. In the following, we will only consider the cases where $n \geq 2$. In this case, this Hopf algebra is the quantum group of the bilinear (but non necessarily symmetric) form defined by the matrix E, in the sense that $\mathcal{B}(E)$ is the universal Hopf algebra such that the bilinear form is a comodule map (for details see [DL]).

If $q \in k^*$ is an invertible element of the ring k, let $E_q \in GL_2(k)$ be the matrix defined by

$$E_q = \begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix}.$$

The Hopf algebra $\mathcal{B}(E_q)$ is isomorphic to the Hopf algebra $\mathcal{O}_q(SL(2))$ (see the definition of $\mathcal{O}_q(SL(2))$ in [K2]).

Let $n, m \geq 1$ be integers and let $E \in GL_n(k)$, $F \in GL_m(k)$ be invertible scalar matrices. Following Bichon [Bi1], we define the algebra $\mathcal{B}(E,F)$ as the k-algebra generated by $n \times m$ variables $z_{ij}, i = 1, \dots, n; j = 1, \dots, m$, submitted to the matrix relations

$$F^{-1} z^t E z = \mathrm{I_m}, \quad z F^{-1} z^t E = \mathrm{I_n},$$

where z is the matrix of generators z_{ij} and $\mathrm{I_m}, \mathrm{I_n}$ are the identity matrices of size m, n respectively. We consider the k-algebra morphism $\delta : \mathcal{B}(E,F) \to \mathcal{B}(E) \otimes \mathcal{B}(E,F)$, defined by

$$\delta(z_{ij}) = \sum_{k=1}^{n} a_{ik} \otimes z_{kj}, \tag{1}$$

for any $i = 1, \dots, n$ and $j = 1, \dots, m$, that endows $\mathcal{B}(E,F)$ with a left $\mathcal{B}(E)$-comodule algebra structure.

In the same way, we have a k-algebra map $\rho : \mathcal{B}(E,F) \to \mathcal{B}(E,F) \otimes \mathcal{B}(F)$ defined by

$$\rho(z_{ij}) = \sum_{k=1}^{n} z_{ik} \otimes b_{kj},$$

where the b_{ij}'s stands for the canonical generators of $\mathcal{B}(F)$. The algebra morphism ρ endows $\mathcal{B}(E,F)$ with a right comodule algebra structure and $\mathcal{B}(E,F)$ is a $\mathcal{B}(E)$-$B(F)$-bicomodule algebra.

Bichon has proved [Bi1, Propositions 3.3, 3.4] that if k is a field and if $\mathrm{Tr}(E^{-1}E^t) = \mathrm{Tr}(F^{-1}F^t)$, then $\mathcal{B}(E,F)$ is a $\mathcal{B}(E)$-$\mathcal{B}(F)$-bi-Galois object. Note that the matrices of form $F^{-1}F^t$ appear in Riehm's work [R] on the classification of bilinear form. Precisely, for any nondegenerate bilinear map $\beta : V \times V \to k$ given by an invertible matrix F, the matrix $\sigma = F^{-1}F^t$ is called the **asymmetry** of β. Over a commutative ring, Bichon's result extends to the following proposition.

Proposition 2. *The canonical map of $\mathcal{B}(E,F)$ considered as a left (resp right) $\mathcal{B}(E)$-comodule algebra (resp $\mathcal{B}(F)$-comodule algebra) is bijective.*

Moreover, if $\mathcal{B}(E,F)$ is k-faithfully flat, it is a $\mathcal{B}(E)$-$\mathcal{B}(F)$-bi-Galois object.

Proof. The proof is the same as for [Bi1, Propositions 3.3, 3.4]. □

Together with Proposition 1, this yields the following corollary.

Corollary 3. *Assume that k is a field. Let E be an invertible matrix and $q \in k^*$ such that $\mathrm{Tr}(E^{-1}E^t) = -q - q^{-1}$, then there is a bijection between $\mathcal{H}_k(\mathcal{B}(E))$ and $\mathcal{H}_k(\mathcal{O}_q(SL(2)))$.*

3. Classification up to isomorphism

The $\mathcal{B}(E)$-comodule algebras $\mathcal{B}(E,F)$ are generic in the following sense.

Theorem 4. *Let k be a PID, $n \geq 2$ an integer, $E \in GL_n(k)$ and Z be a $\mathcal{B}(E)$-Galois object. Then there exist an integer $m \geq 2$ and an invertible matrix $F \in GL_m(k)$ such that* $\mathrm{Tr}(F^{-1}F^t) = \mathrm{Tr}(E^{-1}E^t)$ *and such that Z is isomorphic to $\mathcal{B}(E, F)$ as a $\mathcal{B}(E)$-Galois object.*

Proof. Let Comod-$\mathcal{B}(E)$ be the monoidal category of right $\mathcal{B}(E)$-comodules, with the tensor product $\otimes$ over k, and Mod(k) be the monoidal category of k-modules. Following Ulbrich [U1], [U2] and Schauenburg [Sa3], to any $\mathcal{B}(E)$-Galois object Z, we associate the fibre functor ω_Z : Comod-$\mathcal{B}(E) \to$ Mod(k) defined by

$$\omega_Z(V) = V \square_{\mathcal{B}(E)} Z$$

for any $V \in$ Comod-$\mathcal{B}(E)$. The map $Z \to \omega_Z$ defines a bijective correspondence between the left $\mathcal{B}(E)$-Galois objects (they are by definition faithfully flat) and the exact monoidal functors (= fibre functors) Comod-$B(E) \to$ Mod(k). Moreover, the fibre functor ω_Z sends comodules that are finitely generated projective k-modules to finitely generated projective k-modules (the functor ω_Z preserves the duals). We denote by $\psi_2 : \omega_Z(V) \otimes \omega_Z(V) \to \omega_Z(V \otimes V)$ and $\psi_0 : \omega_Z(k) \to k$ the monoidal isomorphisms. Note that $\psi_2 : (V \square_{\mathcal{B}(E)} Z) \otimes (V \square_{\mathcal{B}(E)} Z) \to (V \otimes V) \square_{\mathcal{B}(E)} Z$ is induced by the multiplication of Z.

The **fundamental comodule of** $\mathcal{B}(E)$, denoted V_E, is the finite free k-module of rank n with basis $(v_1, \dots, v_n)$ and endowed with the $\mathcal{B}(E)$-comodule structure defined by $\delta(v_i) = \sum_{k=1}^n v_k \otimes a_{ki}$ for $1 \leq i \leq n$. The linear map $\beta_E : V_E \otimes V_E \to k$ defined by $\beta_E(v_i, v_j) = E_{ij}$ for $1 \leq i, j \leq n$ is a $\mathcal{B}(E)$-comodule morphism and induces a map

$$\overline{\beta_E} : W \otimes W \xrightarrow{\psi_2} \omega_Z(V_E \otimes V_E) \xrightarrow{\omega_Z(\beta_E)} \omega_Z(k) \xrightarrow{\psi_0} k,$$

where $W = \omega_Z(V_E)$. Since V_E is free of finite rank, W is a finitely generated projective k-module. The base ring k being principal, it implies that W is a free k-module of finite rank, say m.

Set $F_{ij} = \overline{\beta_E}(w_i \otimes w_j)$ for all $1 \leq i, j \leq m$. Writing the elements $(w_j)_{1 \leq j \leq m}$ as elements of $V_E \otimes Z$ and expanding them in the basis $(v_1, \dots, v_n)$ of V_E, we see that there exist $(t_{ij})_{i=1,\dots,n; j=1,\dots,m} \in Z$ such that $w_j = \sum_{i=1}^n v_i \otimes t_{ij}$ for any $j = 1, \dots, m$. Since $(w_j)_{1 \leq i \leq m}$ belong to the cotensor product $V_E \square_{\mathcal{B}(E_q)} Z$, the elements $(t_{ij})_{i=1,\dots,n; j=1,\dots,m}$ satisfy the relations

$$\delta(t_{ij}) = \sum_{k=1}^n a_{ik} \otimes t_{kj} \tag{2}$$

for all $1 \leq i \leq n$ and $1 \leq j \leq m$.

Since the monoidal isomorphism ψ_2 is given by the multiplication of Z, the image of the base $(w_j)_{1 \leq j \leq m}$ by the map $\overline{\beta_E}$ is equal to

$$\begin{aligned}
F_{ij} &= \overline{\beta_E}(w_i \otimes w_j) \\
&= \psi_0 \circ (\beta_E \otimes \mathrm{id}) \circ \psi_2((\textstyle\sum_{k=1}^n v_k \otimes t_{ki}) \otimes (\sum_{l=1}^n v_l \otimes t_{lj})) \\
&= \psi_0 \circ (\beta_E \otimes \mathrm{id})(\textstyle\sum_{k,l=1}^n (v_k \otimes v_l) \otimes t_{ki} t_{lj}) \\
&= \psi_0(\textstyle\sum_{k,l=1}^n E_{kl} \otimes t_{ki} t_{lj}) \\
&= \textstyle\sum_{k,l=1}^n E_{kl} t_{ki} t_{lj}
\end{aligned}$$

for any $1 \leq i, j \leq m$. Putting $T = (t_{ij})_{i=1,\dots,n; j=1,\dots,m}$ and $F = (F_{ij})_{1 \leq i,j \leq m}$, we obtain

$$F = T^t E T. \tag{3}$$

Let us now consider the k-linear map $\nu : k \to V_E \otimes V_E$ defined by

$$\nu(1) = \sum_{i,j=1,\dots,n} E^{-1}_{ij} v_i \otimes v_j,$$

where E^{-1}_{ij} denotes the (i,j)-entry of the inverse matrix E^{-1}. Since this map is a $\mathcal{B}(E)$-comodule morphism, it induces a linear map

$$\bar{\nu} : k \xrightarrow{\psi_0^{-1}} \omega_Z(k) \xrightarrow{\omega_Z(\nu)} \omega_Z(V_E \otimes V_E) \xrightarrow{\psi_2^{-1}} \omega_Z(V_E) \otimes \omega_Z(V_E).$$

Let us compute $\bar{\nu}(1)$. We have

$$\begin{array}{rcl} \bar{\nu}(1) & = & \psi_2^{-1} \circ (\nu \otimes \mathrm{id}) \circ \psi_0^{-1}(1) \\ & = & \psi_2^{-1} \circ (\nu \otimes \mathrm{id})(1 \otimes 1) \\ & = & \psi_2^{-1}(\sum_{i,j=1}^n E^{-1}_{ij}(v_i \otimes v_j) \otimes 1 \\ & = & \sum_{k,l=1}^n E^{-1}_{kl}(v_k \otimes 1) \otimes (v_l \otimes 1). \end{array}$$

Expanding $\bar{\nu}(1)$ in the basis $(w_j)_{1 \leq j \leq m}$ of $\omega_Z(V_E)$, we obtain a matrix $(G_{ij})_{1 \leq i,j \leq m} \in M_m(k)$ such that

$$\bar{\nu}(1) = \sum_{i,j=1}^m G_{ij} w_i \otimes w_j = \sum_{i,j=1}^m \sum_{k,l=1}^n G_{ij}(v_k \otimes t_{ki}) \otimes (v_l \otimes t_{lj})$$

for all $1 \leq i, j \leq m$. Then we have

$$\sum_{k,l=1}^n E^{-1}_{kl}(v_k \otimes 1) \otimes (v_l \otimes 1) = \sum_{i,j=1}^m \sum_{k,l=1}^n G_{ij}(v_k \otimes t_{k,i}) \otimes (v_l \otimes t_{lj}),$$

and then

$$E^{-1}_{kl} = \sum_{i,j=1}^m G_{ij} t_{ki} t_{lj},$$

which we can rewrite as

$$E^{-1} = T G T^t. \tag{4}$$

We now prove $G = F^{-1}$. We have

$$(\beta_E \otimes \mathrm{id}_{V_E}) \circ (\mathrm{id}_{V_E} \otimes \nu) = \mathrm{id}_{V_E}$$

and

$$(\mathrm{id}_{V_E} \otimes \beta_E) \circ (\nu \otimes \mathrm{id}_{V_E}) = \mathrm{id}_{V_E}$$

for β_E and ν. Since ω_Z is monoidal, we obtain

$$(\overline{\beta_E} \otimes \mathrm{id}_W) \circ (\mathrm{id}_W \otimes \bar{\nu}) = \mathrm{id}_W$$

and

$$(\mathrm{id}_W \otimes \overline{\beta_E}) \circ (\bar{\nu} \otimes \mathrm{id}_W) = \mathrm{id}_W$$

for $\overline{\beta_E}$ and $\bar{\nu}$. That is for any basis vector w_i we have

$$\sum_{jk}^{m} F_{ij} G_{jk} w_k = w_i \quad \text{and} \quad \sum_{jk}^{m} G_{jk} F_{ki} w_j = w_i.$$

This implies that the matrix G is the inverse of F. Then Relations (3) and (4) yield the relations

$$F^{-1} T^t E T = \mathrm{I}_m \quad \text{and} \quad T F^{-1} T^t E = \mathrm{I}_n. \tag{5}$$

In the same way, we obtain

$$\beta_E \circ \nu(1) = \mathrm{Tr}(E^{-1} E^t).$$

Since ω_Z is monoidal,

$$\overline{\beta_E} \circ \bar{\nu}(1) = \mathrm{Tr}(F^{-1} F^t)$$

has to be equal to $\mathrm{Tr}(E^{-1}E^t)$. When $\bar{k}$ is a field, Bichon has proved in [Bi1, Section 4] that, under this condition, the algebra $\mathcal{B}_{\bar{k}}(E, F)$ is nonzero. Since our base ring k is a PID, it embeds into a field $\bar{k}$. It is clear that for any invertible matrices E, F, the algebras $\mathcal{B}_k(E, F) \otimes_k \bar{k}$ and $\mathcal{B}_{\bar{k}}(E, F)$ are isomorphic. Therefore, $\mathcal{B}_k(E, F)$ is nonzero provided $\mathrm{Tr}(E^{-1}E^t) = \mathrm{Tr}(F^{-1}F^t)$.

In view of (5) the map

$$\varphi(z_{ij}) = t_{ij},$$

defines an algebra morphism $\varphi : \mathcal{B}(E, F) \to Z$. We claim that φ is an isomorphism of $\mathcal{B}(E)$-Galois objects. First to see that φ is a $\mathcal{B}(E)$-comodule morphism, it is enough to check it on the generators (z_{ij}). The definition of the coaction (1) and relation (2) give

$$(\mathrm{Id} \otimes \varphi) \circ \delta_{\mathcal{B}(E,F)}(z_{ij}) = \sum_{k=1}^{n} a_{ik} \otimes t_{kj} = \delta_Z \circ \varphi(z_{ij})$$

for any $1 \leq i \leq n$ and $1 \leq j \leq m$.

The morphism φ is a morphism of $\mathcal{B}(E)$-comodule algebras, is the identity on the coinvariants elements k of Z, and Proposition 2 ensures that the comodule algebra $\mathcal{B}(E, F)$ has a bijective canonical map. Moreover, Z is a faithfully flat Galois extension of k. Then by [Sn, Remark 3.11] the morphism φ is an isomorphism, and Z and $\mathcal{B}(E, F)$ are isomorphic $\mathcal{B}(E)$-Galois objects.

It remains to prove that the size m of $F \geq 2$. First assume that $m = 1$. Then $W = \omega_Z(V_E) \cong k$. By [Sa1], [Sa3], there is an Hopf algebra K such that Z is a $\mathcal{B}(E)$-K-bi-Galois object. Since there exists an inverse Z^{-1} of Z for the groupoid structure of bi-Galois objects, we have

$$V_E \cong V_E \square_{\mathcal{B}(E)} Z \square_K Z^{-1} \cong k \square_K Z^{-1}.$$

Since Z^{-1} is a Galois object, the image $k \square_K Z^{-1}$ of the trivial comodule of dimension one is the algebra $k \cong (Z^{-1})^{coH}$ of coinvariants. Then the size m of F is equal to one only if the size n of E is one. The same argument proves that m cannot be zero.

□

We now turn to the classification of the Galois objects $\mathcal{B}(E, F)$. The following lemma, implicit in [Bi1], will be useful.

Lemma 5. *Let k be a PID, let $n, m \geq 2$ be integers, and $E \in GL_n(k)$ and $F \in GL_m(k)$ be invertible matrices. Assume that $\mathcal{B}(E,F)$ is a $\mathcal{B}(E)$-$\mathcal{B}(F)$-bi-Galois object and let $\varphi : \text{Comod-}\mathcal{B}(E) \to \text{Comod-}\mathcal{B}(F)$ be the associated monoidal equivalence. Let V_E and V_F be the respective fundamental comodules of $\mathcal{B}(E)$ and $\mathcal{B}(F)$. Then*

$$\varphi(V_E) \cong V_F$$

Proof. Let $w_1, \ldots, w_m$ be the canonical basis of V_F. Then we have a $\mathcal{B}(F)$-colinear map $\theta_F : V_F \to \varphi(V_E)$ defined by

$$\theta_F(w_j) = \sum_{i=1}^{n} v_i \otimes z_{ij}.$$

Similarly, we have a $\mathcal{B}(E)$-colinear morphism $\theta_E : V_E \to \varphi^{-1}(V_F)$ defined by

$$\theta_E(v_i) = \sum_{j=1}^{m} w_j \otimes t_{ji},$$

where the t_{ji}'s are the generators of $\mathcal{B}(F,E)$. It is easy to see that $\varphi(\theta_E) \circ \theta_F$ is the canonical isomorphism $V_F \to \varphi(\varphi^{-1}(V_F))$. We deduce that θ_F and θ_E are monomorphisms and then that θ_F is an isomorphism. $\square$

As an immediate consequence of Lemma 5, we have the following necessary condition for $\mathcal{B}(E)$-Galois objects to be cleft.

Corollary 6. *Let k be a PID and $n, m \geq 2$ be integers, $E \in GL_n(k)$, $F \in GL_m(k)$ and $\mathcal{B}(E,F)$ be a cleft $\mathcal{B}(E)$-Galois object. Then $m = n$.*

Proof. If $\mathcal{B}(E,F)$ is a cleft Galois object, the associated fibre functor is isomorphic as a functor to the forgetful functor and in particular preserves the rank of finite free modules. $\square$

Let us now state our classification result for the extensions $\mathcal{B}(E,F)$.

Theorem 7. *Let k be a PID, n, m_1, m_2 be integers ≥ 2 and $E \in GL_n(k), F_1 \in GL_{m_1}(k), F_2 \in GL_{m_2}(k)$ be invertible matrices such that the algebras $\mathcal{B}(E,F_1)$ and $\mathcal{B}(E,F_2)$ are k-faithfully flat. Then the $\mathcal{B}(E)$-Galois objects $\mathcal{B}(E,F_1)$ and $\mathcal{B}(E,F_2)$ are isomorphic if and only if $m_1 = m_2$ and there exists an invertible matrix $P \in GL_{m_1}(k)$ such that $F_1 = PF_2P^t$.*

Note that, by [R] the bilinear forms associated to F_1 and F_2 are equivalent if and only if the asymmetries of F_1 and F_2 are similar.

Proof. As in the proof of [Bi1, Proposition 2.3], one shows that if $P \in GL_m(k)$, the $B(E)$-comodule algebras $\mathcal{B}(E,F)$ and $\mathcal{B}(E,PFP^t)$ are isomorphic.

Conversely assume that $\mathcal{B}(E,F_1)$ and $\mathcal{B}(E,F_2)$ are k-faithfully flat: then Proposition 2 ensures that $\mathcal{B}(E,F_1)$ and $\mathcal{B}(E,F_2)$ are Galois objects. Let V_E be the fundamental $\mathcal{B}(E)$-comodule and let $\beta_E : V_E \otimes V_E \to k$ be the linear map defined by E. Let $\omega_1 = -\square_{\mathcal{B}(E)}\mathcal{B}(E,F_1)$ and $\omega_2 = -\square_{\mathcal{B}(E)}\mathcal{B}(E,F_2)$ be the fibre functors associated to $\mathcal{B}(E,F_1)$ and $\mathcal{B}(E,F_2)$.

By Lemma 5, the vector space $\omega_1(V_E)$ has a basis $(w_1^1, \ldots, w_{m_1}^1)$ and $\omega_2(V_E)$ has a basis $(w_1^2, \ldots, w_{m_2}^2)$. The comodule algebra isomorphism $\varphi : \mathcal{B}(E,F_1) \to$

$\mathcal{B}(E, F_2)$ induces an isomorphism $\mathrm{id} \otimes \varphi : \omega_1(V_E) \to \omega_2(V_E)$. Then in particular the rank of these two free k-modules is the same, that is $m_1 = m_2 = m$. Let $P \in GL_m(k)$ be the matrix of $\mathrm{id} \otimes \varphi$ in the bases $(w_1^1, \dots, w_{m_1}^1)$ and $(w_1^2, \dots, w_{m_2}^2)$.

The matrices of the bilinear maps $\omega_1(\beta_E)$ and $\omega_2(\beta_E)$, in the bases $(w_1^1, \dots, w_m^1)$ and $(w_1^2, \dots, w_m^2)$, are F_1 and F_2 respectively. Moreover, the isomorphism φ gives the relation

$$\omega_1(\beta_E) = \omega_2(\beta_E) \circ ((\mathrm{id} \otimes \varphi) \otimes (\mathrm{id} \otimes \varphi)).$$

That is for any $i, j = 1, \dots, m$

$$\begin{array}{rcl} \omega_1(\beta_E)(w_i^1 \otimes w_j^1) & = & \omega_2(\beta_E)\left((\sum_{k=1}^m P_{ik} w_k^2) \otimes (\sum_{l=1}^m P_{jl} w_j^2) \right) \\ (F_1)_{ij} & = & \sum_{k,l=1}^m P_{ik} P_{jl} (F_2)_{kl}, \end{array}$$

or in matrix form $F_1 = P F_2 P^t$. □

Remark 8. As an application of Theorem 7, let us consider the case where the matrix F is symmetric. Let k be a PID, let $n, m, p \geq 2$ be integers, and $E \in GL_n(k)$, $F \in GL_m(k)$ and $G \in GL_p(k)$ be invertible matrices. Assume that F is symmetric and $\mathcal{B}(E, F)$ is a Galois object. Then the Galois objects $\mathcal{B}(E, F)$ and $\mathcal{B}(E, G)$ are isomorphic if and only if G is symmetric of size $p = m$.

We now consider the case when k is a field. For any integer $n \geq 2$, and any invertible matrix $E \in GL_n(k)$ we define

$$X_0(E) = \{F \in GL_m(k), m \geq 2, \mathrm{Tr}(F^{-1}F^t) = \mathrm{Tr}(E^{-1}E^t)\}.$$

Consider the equivalence relation $\sim$ defined by $F_1 \sim F_2$ if and only if there exists $P \in GL(k)$ such that $F_1 = P F_2 P^t$ and put $X(E) = X_0(E)/\sim$.

Corollary 9. *Assume that k is a field. Then for any $n \geq 2$ and $E \in GL_n(k)$, there is a bijection $\psi : X(E) \to \mathrm{Gal}(\mathcal{B}(E))$ sending F onto $[\mathcal{B}(E, F)]$.*

Proof. Propositions 3.2, 3.3 and 3.4 in [Bi1] ensure that we have indeed this map ψ. Moreover, ψ is surjective by Theorem 4 and injective by Theorem 7. □

We also have the following result.

Corollary 10. *Assume that k is an algebraically closed field of characteristic zero. For any $n \geq 2$ and $E \in GL_n(k)$, the group of $\mathcal{B}(E)$-$\mathcal{B}(E)$-bi-Galois objects is trivial.*

Proof. Let Z be a $\mathcal{B}(E)$-$\mathcal{B}(E)$-bi-Galois object. By Theorem 4, there exist $m \geq 2$ and $F \in GL_m(k)$ such that Z is isomorphic to $\mathcal{B}(E, F)$ as a $\mathcal{B}(E)$-Galois object. Bichon [Bi1, Propositions 3.3, 3.4] has proved that $\mathcal{B}(E, F)$ is a $\mathcal{B}(E)$-$\mathcal{B}(F)$-bi-Galois object. Then by [Sa1, Theorem 3.5] the Hopf algebras $\mathcal{B}(E)$ and $\mathcal{B}(F)$ are isomorphic that is, by [Bi1, Theorem 5.3], there exists $P \in GL(k)$ such that $F = P^t E P$. The matrix P enables us to construct an isomorphism of left $\mathcal{B}(E)$-Galois objects $Z \cong \mathcal{B}(E, F) \cong \mathcal{B}(E)$. Now since Z is a $\mathcal{B}(E)$-bi-Galois object, we know from [Sa1, Theorem 3.5] that there exists $f \in \mathrm{Aut}(\mathcal{B}(E))$ such that $Z \cong \mathcal{B}(E)^f$ as $\mathcal{B}(E)$-bi-Galois objects. Such a bi-Galois object is trivial if and only if f is coinner. Since by [Bi1, Theorem 5.3] any Hopf automorphism of $\mathcal{B}(E)$ is coinner, we are done. □

The lazy cohomology group of a Hopf algebra was introduced in [BC], where it was realized as a subgroup of the group of bi-Galois objects. Therefore, we have the following.

Corollary 11. *Assume that k is an algebraically closed field of characteristic zero. The lazy cohomology group of $\mathcal{B}(E)$ is trivial for any $E \in GL_n(k)$.*

4. Galois objects up to homotopy

In this section we study the homotopy theory of $\mathcal{B}(E)$-Galois objects. We assume that k is an algebraically closed field. For technical reasons we only consider $\mathcal{O}_q(SL(2))$-Galois objects (recall that $\mathcal{O}_q(SL(2)) = \mathcal{B}(E_q)$). Since for any $E \in GL_n(k)$ there exists a $\mathcal{B}(E)$-$\mathcal{B}(E_q)$-bi-Galois object, Proposition 1 ensures that $\mathcal{H}(\mathcal{B}(E)) \cong \mathcal{H}(\mathcal{B}(E_q))$ and then there is no loss of generality.

We begin, using Lemma 5, by giving a necessary condition for two $\mathcal{B}(E_q)$-Galois extensions to be homotopically equivalent.

Proposition 12. *Let $m_0, m_1 \geq 2$ be integers, let $F_0 \in GL_{m_0}(k)$, $F_1 \in GL_{m_1}(k)$ and assume that $\mathcal{B}(E_q, F_0)$ and $\mathcal{B}(E_q, F_1)$ are $\mathcal{B}(E_q)$-Galois objects. If $\mathcal{B}(E_q, F_0)$ and $\mathcal{B}(E_q, F_1)$ are homotopically equivalent, then the matrices F_0 and F_1 have the same size $m_0 = m_1$.*

Proof. Let us consider two $\mathcal{B}(E_q)$-Galois objects $\mathcal{B}(E_q, F_0)$ and $\mathcal{B}(E_q, F_1)$ with homotopy $\mathcal{B}_{k[t]}(E_q, F_t)$ (by Theorem 4, any $\mathcal{B}_{k[t]}(E_q)$-Galois object is of this form for some $F_t \in GL_m(k[t])$). Then $V_E \square_{\mathcal{B}_{k[t]}(E_q)} \mathcal{B}_{k[t]}(E_q, F_t)$ is a finite free $k[t]$-module of rank equal to the size of the matrix F_t, which does not depend on t. The evaluation at $t = 0, 1$ gives $m_0 = m_1$. $\square$

Let us state a sufficient condition for two $\mathcal{B}(E)$-Galois objects to be homotopically equivalent.

Theorem 13. *Let k be an algebraically closed field, $m_0, m_1 \geq 2$ be integers and F_0, F_1 be invertible matrices of size m_0, m_1 such that $\mathrm{Tr}(F_i^{-1}F_i^t) = -q - q^{-1}$ for $i = 0, 1$.*

If $m_0 = m_1$ and if $F_0^{-1}F_0^t$ and $F_1^{-1}F_1^t$ have the same characteristic polynomial, then the two $\mathcal{O}_q(SL(2))$-Galois objects $\mathcal{B}(E_q, F_0)$ and $\mathcal{B}(E_q, F_1)$ are homotopically equivalent.

The rest of the section is devoted to the proof of the theorem. To this end, we construct a homotopy between the Galois objects, that is an $\mathcal{O}_q(SL(2))[t]$-Galois object over the polynomial ring $k[t]$. First, let us begin with some terminology. We will say that a matrix $F \in GL_m(k)$ (here k is an arbitrary commutative ring) is **manageable** if $F_{mm}^{-1} = 0$ and if the rightmost nonzero coefficient F_{mv}^{-1} in the bottom row is an invertible element of k. In the case of a manageable matrix, the proof of [Bi1, Proposition 3.4] still works and we obtain:

Proposition 14. *Assume that k is a commutative ring and let $F \in GL_m(k)$ be a manageable matrix such that $\mathrm{Tr}(F^{-1}F^t) = -q - q^{-1}$. Then $\mathcal{B}(E_q, F)$ is a free k-module.*

The problem for constructing an homotopy is the following one.

(P) *Let* $F_0, F_1 \in GL_m(k)$ *be manageable matrices such that* $\mathrm{Tr}(F_0^{-1}F_0^t) = \mathrm{Tr}(F_1^{-1}F_1^t)$. *Find a matrix* $F(t) \in GL_m(k[t])$ *such that*

(1) $F(0) = F_0,\ F(1) = F_1$.
(2) $\mathrm{Tr}(F(t)^{-1}F(t)^t) = \mathrm{Tr}(F_0^{-1}F_0^t) = \mathrm{Tr}(F_1^{-1}F_1^t)$.
(3) $F(t)$ *is manageable.*

Now assume that F_0 and F_1 have diagonal block decompositions with the same size:

$$F_0 = \left(\begin{array}{c|c} (F_0)_{11} & 0 \\ \hline 0 & (F_0)_{22} \end{array}\right), \quad F_1 = \left(\begin{array}{c|c} (F_1)_{11} & 0 \\ \hline 0 & (F_1)_{22} \end{array}\right),$$

that

$$\mathrm{Tr}((F_0)_{11}^{-1}(F_0)_{11}^t) = \mathrm{Tr}((F_1)_{11}^{-1}(F_1)_{11}^t)$$

and

$$\mathrm{Tr}((F_0)_{22}^{-1}(F_0)_{22}^t) = \mathrm{Tr}((F_1)_{22}^{-1}(F_1)_{22}^t)$$

and finally that each block is manageable. Then clearly Problem (P) reduces to the same problem for each block. This simple remark, combined with Riehm's work [R] on the structure of bilinear forms, will reduce our problem to the case of some "elementary" matrices.

We will use freely the following results of [R]. For any nondegenerate bilinear map $\beta : V \times V \to k$ given by an invertible matrix F, and for any eigenvalue $p \neq \pm 1$ of its asymmetry σ, p^{-1} is also an eigenvalue of σ and the two characteristic spaces C_p and $C_{p^{-1}}$ associated to p and p^{-1} are isotropic (for the bilinear form β). The vector space V is the orthogonal sum of the subspaces C_1, C_{-1} and $C_p \oplus C_{p^{-1}}$, where p runs over all eigenvalues of σ different from ± 1. Then there exists a basis of V such that the matrix of σ is a block matrix made of Jordan blocks of odd dimension with eigenvalue 1, Jordan blocks of even dimension with eigenvalue -1 and pairs of blocks of eigenvalues p, p^{-1} and of the same dimension.

Assume that the asymmetries σ_0 and σ_1 associated to F_0 and F_1 have the same characteristic polynomial and that σ_1 is diagonal. Then by [R], Problem (P) reduces to three cases.

A. σ_0 is a Jordan block of even dimension d with eigenvalue -1 (and $\sigma_1 = -\mathrm{I}_d$),
B. σ_0 is a Jordan block of odd dimension d with eigenvalue 1 (and $\sigma_1 = \mathrm{I}_d$),
C. σ_0 is a diagonal block matrix made of two Jordan blocks of eigenvalues p, p^{-1} and of the same size d (and σ_1 is diagonal with d diagonal coefficients equal to p and d equal to p^{-1}).

Let us now look at the possible forms of a matrix F such that $\sigma_0 = F^{-1}F^t$ for each of these three cases.

Lemma 15. *A) If* σ_0 *is a Jordan block of even dimension with eigenvalue equal to* -1 *and if there exists an invertible matrix* F *such that* $F^{-1}F^t = \sigma_0$*, then* F

has the lower anti-triangular form

$$F = \begin{pmatrix} 0 & \cdots & \cdots & 0 & F_{1n} \\ \vdots & & \iddots & -F_{1n} & * \\ \vdots & \iddots & \iddots & * & * \\ 0 & F_{1n} & * & * & * \\ -F_{1n} & * & * & * & * \end{pmatrix}. \tag{6}$$

B) If σ_0 is a Jordan block of odd dimension with eigenvalue equal to 1 *and there exists an invertible matrix F such that $F^{-1}F^t = \sigma_0$, then F has the lower anti-triangular form*

$$F = \begin{pmatrix} 0 & \cdots & \cdots & 0 & F_{1n} \\ \vdots & & \iddots & -F_{1n} & * \\ \vdots & \iddots & \iddots & * & * \\ 0 & -F_{1n} & * & * & * \\ F_{1n} & * & * & * & * \end{pmatrix}. \tag{7}$$

C) If σ_0 is made of two Jordan blocks of eigenvalues p and p^{-1} and of size n, then the invertible matrix F defined by

$$\begin{pmatrix} 0 & \mathrm{I_n} \\ \mathrm{J_p} & 0 \end{pmatrix}, \tag{8}$$

where $\mathrm{I_n}$ *is the identity of size n,* $\mathrm{J_p}$ *is a Jordan block of size n and eigenvalue p and* 0 *is the zero matrix, has an asymmetry similar to σ_0.*

Proof. We say that the elements $a_{i,n+1-i}$, for $1 \leq i \leq n$ of a matrix $A \in \mathrm{M}_n(k)$ lies on the anti-diagonal and we use obvious notion of lower and upper anti-triangular matrices.

A) Assume that F is a matrix such that $F^{-1}F^t = \sigma_0$, that is $F^t = F\sigma_0$ or

$$\left\{ \begin{array}{lcll} F_{i1} & = & -F_{1i} & \forall i = 1, \ldots, n \\ F_{ji} & = & F_{i,j-1} - F_{ij} & \forall i = 1, \ldots, n; j = 2, \ldots, n \end{array} \right. \tag{9}$$

Let us consider the first row. The equation $F_{11} = -F_{11}$ implies $F_{11} = 0$. Then, for any $k = 2, \ldots, n$, we have from (9) the equations $F_{1k} = -F_{k1}$ and $F_{k1} = F_{1,k-1} - F_{1k}$ and then $F_{1,k-1} = 0$. Then the first row and the first column are equal to zero except the last terms F_{1n} and F_{n1}.

For the second row and column, we have from the previous computation $F_{12} = F_{21} = 0$ then $F_{22} = F_{21} - F_{22} = 0$. For any $k = 3, \ldots, n-1$ we have

$$\left\{ \begin{array}{lcl} F_{2k} & = & F_{k1} - F_{k2} \\ F_{k2} & = & F_{2,k-1} - F_{2k}. \end{array} \right.$$

Then $F_{2,k-1} = 0$ and, since $F_{2,k-1} = F_{k-1,1} - F_{k-1,2}$, we also have $F_{k-1,2} = 0$. Then for all $k \leq n-2$ the entries $F_{2,k}$ and $F_{k,2}$ are equal to zero. In the same way, any coefficient lying above the anti-diagonal is equal to zero.

The coefficient $F_{i,n+1-i}$ on the anti-diagonal satisfies the relation $F_{i,n+1-i} = F_{n+1-i,i-1} - F_{n+1-i,i} = -F_{n+1-i,i}$. We also have

$$\left\{ \begin{array}{l} F_{i,n+2-i} = F_{n+2-i,i-1} - F_{n+2-i,i} \\ F_{n+2-i,i} = F_{i,n+1-i} - F_{i,n+2-i}, \end{array} \right. \tag{10}$$

then

$$F_{n+2-i,i} = F_{i,n+1-i} - F_{n+2-i,i-1} + F_{n+2-i,i} \tag{11}$$

that is

$$F_{i,n+1-i} = F_{n+1-(i-1),(i-1)}. \tag{12}$$

Then the determinant of F is $(F_{1n})^n$ and the matrix has the wanted form.

B) Let us now consider the case where σ_0 is a Jordan block of odd dimension and eigenvalue 1 and F is a matrix such that $F^{-1}F^t = \sigma_0$, that is $F^t = F\sigma_0$ or

$$\begin{cases} F_{1i} & = & F_{i1} & \forall i = 1, \dots, n \\ F_{ji} & = & F_{i,j-1} + F_{ij} & \forall i = 1, \dots, n; j = 2, \dots, n \end{cases} \tag{13}$$

Let us consider the first line. For any $k = 2, \dots, n$ we have from (13) the equations $F_{1k} = F_{k1}$ and $F_{k1} = F_{1,k-1} + F_{1k}$ and then $F_{1,k-1} = 0$, since $F_{1,k-1} = F_{k-1,1}$, the first line and the first column are equal to zero except the last terms $F_{1n} = F_{n1}$. In the same way as for the previous case, we see that all the coefficients lying above the anti-diagonal must be zero. Moreover, in the same way as for (10) - (12), the anti-diagonal coefficient in position $(i, n-i+1)$ is $(-1)^{i+1}F_{1n}$; the determinant is $(F_{1n})^n$ and F has the wanted form.

C) Assume that σ_0 is made of two Jordan blocks of eigenvalues p and p^{-1} and of size n. We define F by the relation (8). Its asymmetry is the matrix

$$\begin{pmatrix} \mathrm{J_p}^{-1} & 0 \\ 0 & \mathrm{J_p}^t \end{pmatrix}$$

which is similar to σ_0. □

Proof of Theorem 13. Let us construct the matrix $F(t)$ solution of the problem (P).

Cases A, B: Let us consider a Jordan block σ_0 with eigenvalue ± 1 and size more than two. By the previous lemma 15, the matrix F such that $F^{-1}F^t = \sigma_0$ is an anti-triangular matrix of form (6) or (7). Consider the matrix $F(t) \in GL_n(k[t])$ defined by

$$F(t)_{i,n+1-i} = F_{i,n+1-i}, \quad F(t)_{ij} = tF_{ij},$$

for any $1 \leq i, j \leq n$ such that $j \neq n+1-i$ (that is $F(t)$ is equal to F on the anti-diagonal and to tF on the other coefficients).

To compute $\mathrm{Tr}(F(t)^{-1}F(t)^t)$ we have to know the diagonal coefficients of the asymmetry of $F(t)$, which are equal to products of the anti-diagonal coefficients of $F(t)^{-1}$ and $F(t)^t$. Remark that if a matrix $F(t)$ is lower anti-triangular, its inverse $F(t)^{-1}$ is upper anti-triangular, and their anti-diagonal coefficients are related by

$$1 = (F(t)^{-1})_{i,n+1-i}(F(t))_{n+1-i,i},$$

for any $i = 1, \dots, n$. Since the anti-diagonal coefficients of $F(t)$ do not depend on t, the ones of $F(t)^{-1}$ do not depend on t either and we have

$$\mathrm{Tr}(F(t)^{-1}F(t)^t) = \mathrm{Tr}(F_0^{-1}F_0^t) = \mathrm{Tr}(F_1^{-1}F_1^t).$$

From the definition of $F(t)$, we have $F(0) = F_0$ and $F(1)$ is a block matrix with anti-diagonal blocks. Then the asymmetry of $F(1)$ is diagonal and then equal to σ_1.

Since $F(t)^{-1}$ is upper anti-triangular and invertible, $F(t)$ is manageable. Finally, $F(t)$ is a solution of (P) in the cases A,B.

Case C: Let us now consider the case of two Jordan blocks of size n and eigenvalues p and p^{-1} and suppose that F has the form (8).

Consider the matrix $F(t) \in GL_{2n}(k[t])$ defined by

$$\begin{pmatrix} 0 & \mathrm{I_n} \\ \mathrm{J_p(t)} & 0 \end{pmatrix},$$

where $\mathrm{J_p(t)}$ is the matrix with diagonal coefficients equal to p and upper diagonal coefficients $(i, i+1)$ equal to t. The inverse $F(t)^{-1}$ is

$$\begin{pmatrix} 0 & (\mathrm{J_p(t)})^{-1} \\ \mathrm{I_n} & 0 \end{pmatrix}$$

and then $F(t)$ is manageable and the trace of its asymmetry is constant. Finally $F(t)$ is a solution of (P) in the case C. □

Corollary 16. *All cleft $\mathcal{O}_q(SL(2))$-Galois objects are homotopically trivial.*

Proof. Let Z be a cleft Galois object of $\mathcal{O}_q(SL(2))$. Then by Theorem 4, the Galois object Z is isomorphic to $\mathcal{B}(E_q, F)$ and by Corollary 6 the matrix F is a 2×2 matrix with trace equal to $-q - q^{-1}$.

If $q \neq 1$, there exists $P \in GL(k)$ such that $F = PE_qP^t$. Then the Galois object $\mathcal{B}(E_q, F)$ is isomorphic to the trivial object $\mathcal{B}(E_q)$.

If $q = 1$, the two possible asymmetries are, up to similarity, a diagonal matrix σ_1 with eigenvalue -1 and multiplicity 2 associated to a matrix F_1 or a Jordan block matrix σ_2 of size 2 and eigenvalue -1 associated to a matrix F_2. The two associated Galois objects $\mathcal{B}(E_q, F_1)$ and $\mathcal{B}(E_q, F_2)$ are nonisomorphic as the asymmetries are nonsimilar, but they are homotopically equivalent by Theorem 13 as the asymmetries have the same characteristic polynomial. □

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